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Discussion

Comments on "Generalized spectral decomposition and intrinsic irreversibility of the Arnold Cat Map" by I. Antoniou, B. Qiao and Z. Suchanecki [Chaos, Solitons & Fractals 8 (1997) 77–90]

An understanding of the character of the Frobenius-Perron and Koopman operators of classical dynamical systems (which describe the evolution of the densities and system observables, respectively) provides deep insights into the nature of the system dynamics. In particular, the spectral decomposition of these operators enables one to understand the decay properties of all correlation functions and the computation of system transport properties. Of particular interest are studies of chaotic systems insofar as they promise to provide deep connections between chaos and statistical mechanics.

The spectral decomposition of the Frobenius-Perron and Koopman operators in rigged Hilbert spaces has been achieved for a small number of Kolmogorov and exact systems. Antoniou, Qiao and Suchanecki (denoted AQS) recently published a generalized spectral decomposition of these operators for the important case of the Arnold Cat Map [1]. Unfortunately, as shown below, the proposed decomposition is incorrect.

The essential problem is that the generalized left and right eigenvectors ($(\varphi^k_{m'n'}|$ and $|\Phi^l_{mn})$ respectively), constructed to diagonalize the Koopman operator V, do not do so. To see this consider, for example, the following computation, where we work in the $\bar{\Omega}$ space and use the notation introduced by AQS.

$$(\varphi^1_{m'n'}|V|\Phi^2_{mn}) = \frac{1}{\sqrt{5}} \int \mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\; \varphi^1_{m'n'}(\bar{x},\bar{y}) V \Phi^2_{mn}(\bar{x},\bar{y}) \tag{1}$$

$$= \frac{1}{\sqrt{5}} \frac{(-1)^{m'}}{m'!} \int_{\bar{D}_1} \mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\; \bar{y}\delta^{(m')}(\bar{y})\bar{x}^{n'} V \Phi^2_{mn}(\bar{x},\bar{y}) \tag{2}$$

$$= \frac{1}{\sqrt{5}} \frac{(-1)^{m'}}{m'!} \int_{\bar{D}_1} \mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\; \bar{y}\delta^{(m')}(\bar{y})\bar{x}^{n'} \Phi^2(\lambda\bar{x},\xi\bar{y}) \tag{3}$$

$$= \frac{(-1)^{m'+n}}{m'!n!} \int_{\bar{D}_1} \mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\; \bar{y}\delta^{(m')}(\bar{y})\bar{x}^{n'} (\xi\bar{y}+\Phi)^{m-1} \delta^{(n)}\left(\lambda\bar{x} - \frac{1}{\Phi}\right) \mathbf{1}_{\bar{D}_2}(\lambda\bar{x},\xi\bar{y}). \tag{4}$$

Here we have used Eq. (26) of Ref. [1] to compute the action of V in Eq. (2), yielding Eq. (3).

This integral is non-zero iff $(\bar{x},\bar{y}) = (1/\lambda\phi, 0) \in \bar{D}_1$ and $(\lambda\bar{x},\xi\bar{y}) = (1/\phi, 0) \in \bar{D}_2$. This condition does indeed hold. To see this visually note that

$$\lambda\phi = \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{5}+1}{2}\right) = 2+\sqrt{5},$$

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so that

$$\frac{1}{\lambda\phi} = \frac{1}{2+\sqrt{5}} < \frac{1}{2}.$$

By inspection of Fig. 5 of Ref. [1], this is seen to be in the set $\bar{D}_1$. It is equally clear from the same figure that $(1/\phi, 0)$ is in the set $\bar{D}_2$.

We may therefore write Eq. (1) as

$$(\varphi^1_{m'n'}|V|\Phi^2_{mn}) = \frac{\xi^{m+n}(-1)^{m'+n}}{m'!n!}\left[\frac{\partial^{m'}}{\partial\bar{y}^{m'}}\{\bar{y}(\bar{y}+\lambda\phi)^{m-1}\}\right]_{\bar{y}=0}\cdot\left[\frac{\partial^n}{\partial\bar{x}^n}\{\bar{x}^{n'}\}\right]_{\bar{x}=1/\lambda\phi}. \tag{5}$$

The term with derivatives in $\bar{x}$ equals

$$\left[\frac{\partial^n}{\partial\bar{x}^n}\{\bar{x}^{n'}\}\right]_{\bar{x}=1/\lambda\phi} = \begin{cases} 0, & n > n', \\ n!, & n = n', \\ \dfrac{n'!}{(n'-n)!}(\lambda\phi)^{-n'-n}, & n < n' \end{cases} \tag{6}$$

and the term with derivatives in $\bar{y}$ equals

$$\left[\frac{\partial^{m'}}{\partial\bar{y}^{m'}}\{\bar{y}(\bar{y}+\lambda\phi)^{m-1}\}\right]_{y=0} = \begin{cases} 0, & m' > m, \\ m!, & m = m', \\ \dfrac{m'(m-1)!}{(m-m')!}(\lambda\phi)^{m-m'}, & m' < m. \end{cases} \tag{7}$$

Thus, $(\varphi^1_{m'n'}|V|\Phi^2_{mn})$ is in general non-zero and this basis does not diagonalize the Koopman operator V. This result arises directly from the effect of V on the characteristic functions $\mathbf{1}_{\bar{D}_k}$, an effect which was neglected in Ref. [1].

Hence we conclude that the spectral decomposition in Ref. [1], the central result of the paper, is incorrect. The author's response (below) resolves the original problem but introduces an incomplete basis which is therefore unacceptable.

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References

[1] Antoniou I, Qiao B, Suchanecki Z. Chaos, Solitons and Fractals 1997;8:77.

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