Generation and Control of Chains of Entangled Atom-Ion Pairs with Quantum Light

Moshe Shapiro[1,2] and Paul Brumer[3]

[1]Department of Chemistry, University of British Columbia, Vancouver, British Columbia V6T 1Z3, Canada
[2]Department of Chemical Physics, The Weizmann Institute, Rehovot 76100, Israel
[3]Chemical Physics Theory Group, Department of Chemistry, University of Toronto, Toronto, Ontario M5S 3H6, Canada

(Received 2 December 2010; published 14 April 2011)

Coherent control using quantum light incident upon molecules in an optical lattice is shown to give rise to a direct way of writing arbitrary sequences of entangled atom-ion pairs. There is no evident limitation on the length of the word (i.e., the number of qbits) that can be formed.

DOI: 10.1103/PhysRevLett.106.150501 PACS numbers: 03.67.Bg, 03.65.Ud, 32.80.−t, 42.50.Ex

Entanglement is a central attribute of quantum mechanics and widely regarded as a fundamental resource for quantum computing, quantum dense coding, and quantum teleportation [1–4]. In this Letter we significantly extend the range of possible entangled states by describing a general scenario for preparing entangled states comprising sequences of atom-ion pairs entangled with coherent states of light. The proposed method, based on coherent control techniques, relies upon available technologies and offers the opportunity to write arbitrary length sequences of atom-ion pairs entangled with radiation field states whose content is controlled by well-defined external fields.

Note first that coherent control of directionality in dissociation using classical electromagnetic radiation has been developed theoretically and implemented experimentally [5–8]. Below we propose the significant extension to quantum light, with the specific goal noted above. To expose the origin of the symmetry breaking in this approach, and introduce notation, we first consider control with coherent states and show that it is, as expected, identical to coherent control with classical fields [8].

Consider a radiation field composed of a product of coherent states, $|\alpha_\omega\rangle$, defined [9,10] as $\hat{a}_\omega|\alpha_\omega\rangle = \alpha_\omega|\alpha_\omega\rangle$, where $\hat{a}_\omega$ is the annihilation operator of a photon in mode ω. We consider only two field modes, that of ω and that of $\omega/2$; hence the initial matter + radiation state is

$$|\Psi^i\rangle = |\alpha_\omega, \alpha_{\omega/2}\rangle|E_i\rangle, \tag{1}$$

where $|E_i\rangle$ is some bound material state of energy E_i. Because $|\alpha_\omega\rangle$ is an eigenstate of the annihilation operator, which is the only part of the quantum field operator that contributes in the rotating wave approximation, the absorption of one photon from an $|\alpha_\omega\rangle$ state results in the production of $\bar{\epsilon}(\omega)|\alpha_\omega\rangle$, the initial coherent state multiplied by the field amplitude $\bar{\epsilon}(\omega) = [-\hbar\omega/(\varepsilon_0 V)]^{1/2}\alpha_\omega \exp(-i\omega z/c - i\phi_\omega)$, with z denoting the axis of propagation of the light beams, ϵ_0 being the permittivity of the vacuum, V, the cavity volume, and ϕ_ω is an extra phase which we add (by, e.g., modifying the optical path transversed by the field) to the inherent phase associated with the time dependence of the quantum state of the radiation. As in classical-field coherent control, a $\phi_\omega - 2\phi_{\omega/2}$ phase difference is introduced in order to control processes, as described below.

Given the initial state in Eq. (1), the final matter + radiation state is [8],

$$|\Psi^f\rangle = (2\pi i)^{1/2}|E, \mathbf{m}^-\rangle\{\bar{\epsilon}(\omega)\langle E, \mathbf{m}^-|d|E_i\rangle - \bar{\epsilon}^2(\omega/2)\langle E, \mathbf{m}^-|D|E_i\rangle\}|\alpha_\omega, \alpha_{\omega/2}\rangle. \tag{2}$$

where $|E, \mathbf{m}^-\rangle$ are incoming continuum states characterized by energy $E = E_i + \hbar\omega$ and remaining quantum number $\mathbf{m}$, d is the dipole operator $\mathbf{d}$ projected along the polarization direction $\hat{\epsilon}$ and D is the operator (see, e.g., [8]) inducing two-photon absorption via the $\omega/2$ field that is also polarized along $\hat{\epsilon}$. Equation (2) makes clear that in this case the radiation field does not get entangled with the material states. Hence, tracing over the radiation states to expose the system subspace of interest [11,12] does not destroy the matter superposition state in Eq. (2), whose content can be controlled to enhance production of selected product states [5–7]. We see that achieving coherent control with coherent states via this "1 vs 2 photon" scenario is possible and essentially identical to the classical-field case.

We now show that when one uses nonclassical light [13,14] it is possible to create light-matter entangled states [12,15–19], and specifically, to create an arbitrary sequence of mutually entangled atom-ion pairs, all of which are entangled to the light states. In order to achieve this goal we consider the use of "cat states" [20–22] of the radiation field, defined as

$$|\mathcal{C}\rangle \equiv \frac{1}{\sqrt{2}}(|\alpha_\omega\rangle + |-\alpha_\omega\rangle). \tag{3}$$

With this choice, the initial matter + radiation state in the 1 vs 2 coherent control scenario is given by

$$|\Psi^i\rangle = \frac{1}{\sqrt{2}}(|\alpha_\omega\rangle + |-\alpha_\omega\rangle)|\alpha_{\omega/2}\rangle|E_i\rangle. \tag{4}$$

Our objective is to control the direction of motion of photofragments, defined by the momentum $\hat{\mathbf{k}}$. Using the rotating wave approximation, the final (continuum) state $|\Psi^f\rangle$ now becomes

 © 2011 American Physical Society

$$|\Psi^f\rangle = (i\pi)^{1/2}|\alpha_{\omega/2}\rangle\sum_{\hat{\mathbf{k}}}|E,\hat{\mathbf{k}}^-\rangle\cdot$$
$$\{[\bar{\epsilon}(\omega)\langle E,\hat{\mathbf{k}}^-|d|E_i\rangle - \bar{\epsilon}^2(\omega/2)\langle E,\hat{\mathbf{k}}^-|D|E_i\rangle]|\alpha_\omega\rangle$$
$$+ [-\bar{\epsilon}(\omega)\langle E,\hat{\mathbf{k}}^-|d|E_i\rangle - \bar{\epsilon}^2(\omega/2)$$
$$\times\langle E,\hat{\mathbf{k}}^-|D|E_i\rangle]|E_i\rangle]| - \alpha_\omega\rangle\}, \quad (5)$$

where each $|E,\hat{\mathbf{k}}^-\rangle$ state is an "incoming" scattering state that asymptotically approaches a free plane wave moving in the $\hat{\mathbf{k}}$ direction.

Evident from Eq. (5) is that this choice of an initial cat state has produced a matter + radiation entangled final state, a consequence of the different signs resulting from the annihilation operator operating on $|\alpha_\omega\rangle$ vs $|\alpha_{-\omega}\rangle$. It is worth emphasizing that this effect relies entirely upon the use of quantum light, since, in the classical limit, the two components in Eq. (3) add together to produce a field of zero amplitude.

To see the entanglement more clearly we rewrite Eq. (5)

$$|\Psi^f\rangle = (i\pi)^{1/2}|\alpha_{\omega/2}\rangle\sum_{\hat{\mathbf{k}}}|E,\hat{\mathbf{k}}^-\rangle\cdot$$
$$\{\bar{\epsilon}(\omega)\langle E,\hat{\mathbf{k}}^-|d|E_i\rangle[|\alpha_\omega\rangle - |-\alpha_\omega\rangle] - \bar{\epsilon}^2(\omega/2)$$
$$\langle E,\hat{\mathbf{k}}^-|D|E_i\rangle[|\alpha_\omega\rangle + |-\alpha_\omega\rangle]\}. \quad (6)$$

Noting that the two-photon amplitude $\langle E,\hat{\mathbf{k}}^-|D|E_i\rangle$ is symmetric about coordinate inversion (i.e., $\hat{\mathbf{k}}_f \leftrightarrow \hat{\mathbf{k}}_b$) where f denotes the forward direction and b denotes the backward direction, and the one-photon amplitude $\langle E,\hat{\mathbf{k}}^-|d|E_i\rangle$ is antisymmetric with respect to this inversion, we can write the product of the one-photon and two-photon amplitudes as

$$\langle E,\hat{\mathbf{k}}^-|D|E_i\rangle\langle E,\hat{\mathbf{k}}^-|d|E_i\rangle = |B(E,\hat{\mathbf{k}}|E_i)|\exp(i\beta),$$

with $|B(E,\hat{\mathbf{k}}_f|E_i)| = |B(E,\hat{\mathbf{k}}_b|E_i)|$ and $\beta \equiv \beta_{\hat{\mathbf{k}}_f} = \pi - \beta_{\hat{\mathbf{k}}_b}$.

To simplify the discussion we now assume that only the forward "f" or backward "b" directions are allowed. This can be achieved experimentally, for example, in a one dimensional array of (ultra)cold molecules (say Ca_2^+ or K_2^+ [6], or K-Rb [23]) trapped in a two dimensional optical lattice. As illustrated in Fig. 1, a pulse polarized in the x direction, composed of an ω cat state and an $\omega/2$ coherent state is allowed to propagate along the array's direction, denoted z, and to dissociate the molecules it encounters as it moves from a z_i site to the next site at z_{i+1}. Each dissociation event yields forward moving fragments or backward moving fragments depending on the imposed phase β. For, e.g., a K_2^+ molecule whose center of mass is located at z_i, the forward direction is defined as the K moving to a neighboring site at $[z_i, \delta x]$, and the K^+ moving to the opposite $[z_i, -\delta x]$ site. The backward direction is that obtained by interchanging the K and the K^+ positions. Clearly, in the K_2^+ case one simply controls the direction of motion of the valence electron as the K_2^+ molecule dissociates [6].

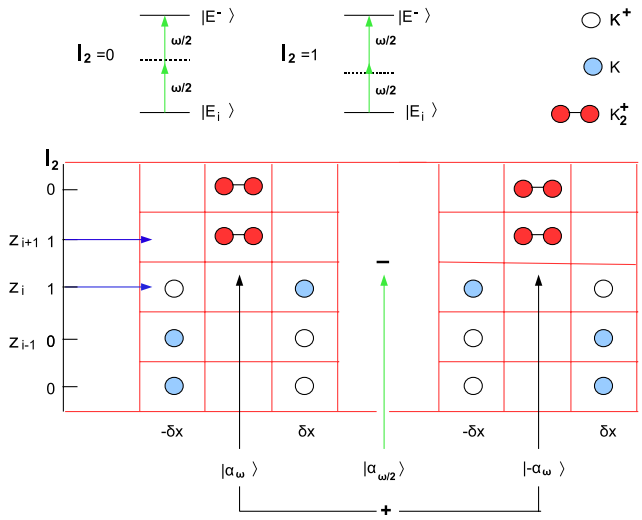

FIG. 1 (color online). Proposed optical lattice setup: The lower panel shows the method of writing an arbitrary word composed of a sequence of entangled qbits using a set of K_2^+ molecules placed along a line in an optical lattice, irradiated by a combination of a $|\alpha_\omega\rangle + |-\alpha_\omega\rangle$ and $|\alpha_{\omega/2}\rangle$ field states. Upper panel: The modulation of the intensity I_2 of an auxiliary laser induces an ac Stark shift of the intermediate resonance of the two-photon dissociation process. Setting the auxiliary laser to off ($I_2 = 0$) or on ($I_2 = 1$) shifts the phase of the two-photon matrix element by π, changing the direction of dissociation.

Note that to ensure that the dissociated particles occupy sites adjacent by δx and $-\delta x$, the lattice should be constructed so that the $2\delta x$ wide region around the initial sequence of molecules is enclosed by a higher optical potential. Further, δx should be chosen so that the final ion-atom pair is in a region sufficiently wide to support an effective translational continuum.

We can further simplify the expressions by tuning the field intensities such that

$$|\epsilon(\omega)/\epsilon^2(\omega/2)| = |\langle E^-|D|E_i\rangle/\langle E^-|d|E_i\rangle|. \quad (7)$$

(Here and below we suppress the $\mathbf{k}$ in the state labels since the $|\mathbf{k}|$ is determined by the value of E.) With this choice of relative intensities, we have that, at site z

$$|\Psi^f\rangle = (i\pi)^{1/2}e^{-i\omega z/c}|\alpha_{\omega/2}|\epsilon(\omega)\langle E^-|d|E_i\rangle|\{|E,f^-\rangle[(|\alpha_\omega\rangle - |-\alpha_\omega\rangle) - e^{i\gamma}(|\alpha_\omega\rangle + |-\alpha_\omega\rangle)]$$
$$+ |E,b^-\rangle[(|\alpha_\omega\rangle - |-\alpha_\omega\rangle) + e^{i\gamma}(|\alpha_\omega\rangle + |-\alpha_\omega\rangle)]\}, \quad (8)$$

where $\gamma \equiv \phi_\omega - 2\phi_{\omega/2} + \beta$. Here $|E,f^-\rangle$ denotes $|E,\mathbf{k}_f^-\rangle$ and $|E,b^-\rangle$ denotes $|E,\mathbf{k}_b^-\rangle$.

If the phase difference of the fields is chosen such that

$$\phi_\omega - 2\phi_{\omega/2} = -\beta, \tag{9}$$

we have from Eq. (8) that

$$|\Psi^f\rangle = -2(i\pi)^{1/2} e^{-i\omega z/c} |\epsilon(\omega)\langle E^-|d|E_i\rangle| |\alpha_{\omega/2}\rangle \times \{|E, f^-\rangle| - \alpha_\omega\rangle - |E, b^-\rangle|\alpha_\omega\rangle\}. \tag{10}$$

Equation (10) describes a matter + radiation entangled superposition state comprised of a material state of the photofragment directed in the forward direction multiplied by the $|-\alpha_\omega\rangle$ radiative coherent state, and a material state of photofragments going in the backward direction accompanied by the $|\alpha_\omega\rangle$ state. The constitution of the matter + radiation entangled state produced in this scenario is completely controllable. For example, if we choose the field's phase difference such that

$$\phi_\omega - 2\phi_{\omega/2} = \pi - \beta, \tag{11}$$

then the resulting state will be

$$|\Psi^f\rangle = 2(i\pi)^{1/2} e^{-i\omega z/c} |\epsilon(\omega)\langle E^-|d|E_i\rangle| |\alpha_{\omega/2}\rangle \times \{|E, f^-\rangle|\alpha_\omega\rangle - |E, b^-\rangle| - \alpha_\omega\rangle\}. \tag{12}$$

The ability to control the entangled state comes from the 1 photon vs 2 photon interference term that survives the tracing operation over the radiative states because, as Eqs. (10) and (12) and show, a measurement of the radiative state ($|\alpha_\omega\rangle$ vs $|-\alpha_\omega\rangle$) does not provide information as to whether the one-photon or the two-photon pathway was followed. However, as in all two-components entangled states, a measurement of one component (e.g., the radiative one) affects the other (material) component. Thus, for example, for $\alpha > \approx 2$ if we tune the radiative phase difference as in Eq. (9) and measure that the radiation state is, say, $|\alpha_\omega\rangle$, then the photofragments must have recoiled in the backward direction [24]. The important aspect here is that the entanglement can be controlled at will. For example, by tuning the radiative phase difference as in Eq. (11) a measurement of an $|\alpha_\omega\rangle$ radiation state implies that the photofragments must have recoiled in the forward direction.

The treatment given above can be extended to create a sequence of entangled atom-ion pairs. Choosing the phases according to Eq. (9) we first dissociate K_2^+ molecule number 1, thereby creating an entangled $|\Psi_1^f\rangle$ state of Eq. (10) with respect to the first pair of atoms,

$$|\Psi_1^f\rangle = -2(i\pi)^{1/2} e^{-i\omega z_1/c} |\epsilon(\omega)\langle E^-|d|E_{i_1}\rangle| \times \{|E, f_1^-\rangle| - \alpha_\omega\rangle - |E, b_1^-\rangle|\alpha_\omega\rangle\}|\alpha_{\omega/2}\rangle. \tag{13}$$

Here and below, $|E_{i_m}\rangle$ denotes a state of energy E_i of the molecule in the mth site along the z axis. We now apply this $|\Psi_1^f\rangle$ state to a second K_2^+ diatomic in the array for which our initial state is, therefore,

$$|\Psi_2^i\rangle = 2(i\pi)^{1/2} e^{-i\omega z_1/c} |\epsilon(\omega)\langle E^-|d|E_{i_1}\rangle| \times \{-|E, f_1^-\rangle| - \alpha_\omega\rangle + |E, b_1^-\rangle|\alpha_\omega\rangle\}|\alpha_{\omega/2}\rangle|E_{i_2}\rangle. \tag{14}$$

The final state for molecule 1 and molecule 2 thus becomes

$$|\Psi_{1,2}^f\rangle = 4i\pi e^{-i\omega(z_1+z_2)/c} |\epsilon^2(\omega)\langle E^-|d|E_{i_2}\rangle\langle E^-|d|E_{i_1}\rangle| \times \{|E, f_2^-\rangle|E, f_1^-\rangle| - \alpha_\omega\rangle + |E, b_2^-\rangle|E, b_1^-\rangle|\alpha_\omega\rangle\}|\alpha_{\omega/2}\rangle. \tag{15}$$

We have thus entangled two atom-ion pairs and the radiation field. Explicitly, if we detect the radiation field to be in the $|-\alpha_\omega\rangle$ state, then both molecules 1 and 2 must have been dissociated in the forward direction, whereas if we detect the radiation field to be in the $|\alpha_\omega\rangle$ state then molecules 1 and 2 must have been dissociated in the backward direction. Alternatively, if we detect the first K atom to be in site $[z_i, \delta x]$, then the second K atom must also be in site $[z_{i+1}, \delta x]$ and the radiation field must be in the $|-\alpha_\omega\rangle$ state. Moreover, we can, if we wish, interchange between the above $|\alpha_\omega\rangle$ and $|-\alpha_\omega\rangle$ radiation states (or interchange $f \leftrightarrow b$) by choosing the field phases according to Eq. (11).

In a similar manner one can continue to act on more and more molecules in the sequence, creating states in which n atom-ion pairs plus one radiation field are entangled together as,

$$|\Psi_{1,\ldots,n}^f\rangle = (4i\pi)^{n/2} |\alpha_{\omega/2}\rangle e^{-i\omega(z_1+z_2+\ldots+z_n)/c} |\epsilon^n(\omega)\Pi_{m=1}^n \langle E^-|d|E_{i_m}\rangle|\{[(-1)^n \Pi_{m=1}^n |E, f_m^-\rangle]| - \alpha_\omega\rangle + [\Pi_{m=1}^n |E, b_m^-\rangle]|\alpha_\omega\rangle\}. \tag{16}$$

Alternatively, as shown in Fig. 1 the phase β can be varied from site to site using an auxiliary classical field which induces an ac Stark shift in an intermediate K_2^+ level. If we assume for simplicity that the two-photon amplitude $\langle E, \hat{\mathbf{k}}^-|D|E_i\rangle$ is dominated by a single intermediate resonance,

$$\langle E, \hat{\mathbf{k}}^-|D|E_i\rangle \approx \langle E, \hat{\mathbf{k}}^-|d|E_n\rangle\langle E_n|d|E_i\rangle/(E_i + \hbar\omega/2 - E_n), \tag{17}$$

then by shifting the intermediate level position E_n from being above $E_i + \hbar\omega/2$ to being below that value, we effectively add a π phase shift to the two-photon process [25]. Thus, the final state to be written, such as,

$$|\Psi_{1,\ldots,n}^f\rangle = (4i\pi)^{n/2} |\alpha_{\omega/2}\rangle e^{-i\omega(z_1+z_2+\ldots+z_n)/c} |\epsilon^n(\omega)\Pi_{m=1}^n \langle E^-|d|E_{i_m}\rangle|\{(-1)^n| - \alpha_\omega\rangle|E, f_1^-\rangle|E, f_2^-\rangle|E, b_3^-\rangle \ldots + |\alpha_\omega\rangle|E, b_1^-\rangle|E, b_2^-\rangle|E, f_3^-\rangle \ldots\} \tag{18}$$

is controlled by the auxiliary field (via, e.g., an intensity modulating pulse shaper).

The result of this scenario is a sequence of atom-ion pairs entangled with one another as well as with the light field. This constitutes an entirely new entangled species, and a possible resource in quantum information. Explicit implementations of quantum computation with this scenario are being developed.

Several remarks are relevant with respect to the influence of decoherence on this scenario. First, note that internal degrees of freedom (e.g., vibration) of the dissociated K_2^+ are irrelevant since they do not entangle with $|\Psi^f_{1,\ldots,n}\rangle$. That is, both $|f\rangle$ and $|b\rangle$ share the same internal states $|v\rangle$ so that $|v\rangle$ factors out the wave function. As such, tracing over any such degrees of freedom does not cause decoherence in $|\Psi^f_{1,\ldots,n}\rangle$. Second, note that a choice to ignore the states of the radiation field in Eq. (18) is equivalent to tracing $|\Psi^f_{1,\ldots,n}\rangle\langle\Psi^f_{1,\ldots,n}|$ over the radiation states yielding for example, for $n = 3$, that

$$\begin{aligned}\mathrm{Tr}[|\Psi^f_{1,\ldots,n}\rangle\langle\Psi^f_{1,\ldots,n}|] = (4\pi)^n|\epsilon^n(\omega)\Pi^n_{m=1}\langle E^-|d|E_{i_m}\rangle|^2[&(|E,f_1^-\rangle\langle E,f_1^-|)(|E,f_2^-\rangle\langle E,f_2^-|)(|E,b_3^-\rangle\langle E,b_3^-|)\\ &+(|E,b_1^-\rangle\langle E,b_1^-|)(|E,b_2^-\rangle\langle E,b_2^-|)(|E,f_3^-\rangle\langle E,f_3^-|)+(-1)^n\exp(-2\alpha_\omega^2)\{(|E,f_1^-\rangle\langle E,b_1^-|)\\ &\times(|E,f_2^-\rangle\langle E,b_2^-|)(|E,b_3^-\rangle\langle E,f_3^-|)+\mathrm{H.c.}\}]\end{aligned} \tag{19}$$

where H.c. denotes the Hermitian conjugate of the preceding term. Significantly, even if we ignore the radiation states, matter entanglement would persist for small α_ω in terms of the last (cross) term. For larger values [24] of α_ω, this entanglement disappears, producing a product state of superpositions of K-K^+ and K^+-K pairs, which is in itself of great interest.

As for decoherence, we note recent studies [26–28] on the coherence of one body and of many body states in optical lattices that show the feasibility of optimizing lattice parameters to sustain coherence. To this end, computations are under way to ascertain optimal conditions to minimize decoherence in this scenario.

The authors acknowledge support from the Natural Sciences and Engineering Research Council of Canada. M. S. acknowledges support from UBC's Peter Wall Institute for Advanced Science.

[1] See M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[2] D. Bruss and G. Leuchs, *Lectures on Quantum Information* (Wiley, Weinheim, 2007).
[3] *Entanglement and Decoherence*, edited by A. Buchleitner, C. Viviescas, and M. Tiersch (Springer, Berlin, 2010).
[4] For an extensive review of entanglement, see R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[5] G. Kurizki, M. Shapiro, and P. Brumer, Phys. Rev. B **39**, 3435 (1989).
[6] For an example of controlled directional photodissociation of a diatomic ion (here D_2^+), see B. Sheehy, B. Walker, and L. F. DiMauro, Phys. Rev. Lett. **74**, 4799 (1995).
[7] E. Dupont, P. B. Corkum, H. C. Liu, M. Buchanan, and Z. R. Wasilewski, Phys. Rev. Lett. **74**, 3596 (1995).
[8] M. Shapiro and P. Brumer, *Principles of the Quantum Control of Molecular Processes* (John Wiley and Sons, New York, 2003).
[9] E. Schrödinger, Naturwissenschaften **14**, 664 (1926).
[10] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[11] M. Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Springer, New York, 2008).
[12] J. B. Gong and P. Brumer, J. Chem. Phys. **132**, 054306 (2010).
[13] J. Peřina, *Quantum Statistics of Linear and Nonlinear Phenomena* (D. Riedl, Dodrecht, Boston, 1984).
[14] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
[15] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).
[16] T. Opatrný and G. Kurizki, Phys. Rev. Lett. **86**, 3180 (2001).
[17] E. Solano, G. S. Agarwal, and H. Walther, Phys. Rev. Lett. **90**, 027903 (2003).
[18] M. Bayer, P. Hawrylak, K. Hinzer, S. Fafard, M. K. Z. R. Wasilewski, O. Stern, and A. Forchel, Science **291**, 451 (2001).
[19] P. Král, I. Thanopulos, and M. Shapiro, Phys. Rev. A **72**, 020303 (2005).
[20] E. Schrödinger, Naturwissenschaften **23**, 807 (1935); **23**, 823 (1935); **23**, 844 (1935).
[21] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[22] W. Schleich, M. Pernigo, and F. L. Kien, Phys. Rev. A **44**, 2172 (1991).
[23] K.-K. Ni, S. Ospelkaus, M. H. G. de Miranda, A. Pe'er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, Science **322**, 231 (2008).
[24] Coherent states are not orthogonal. However, for $\alpha_\omega > \approx 2$, one can regard $|\alpha_\omega\rangle$ and $|-\alpha_\omega\rangle$ as effectively orthogonal since $|\langle-\alpha_\omega|\alpha_\omega\rangle| = \exp(-2|\alpha_\omega|^2)$. Hence, for $\alpha_\omega > \approx 2$, $|\langle-\alpha_\omega|\alpha_\omega\rangle| < 3.4 \times 10^{-4}$. The choice of the α_ω cutoff value is, of course, discretionary.
[25] See, e.g., F. Wang, C. Chen, and D. S. Elliott, Phys. Rev. Lett. **77**, 2416 (1996).
[26] F. Gerbier and Y. Castin, Phys. Rev. A **82**, 013615 (2010).
[27] H. Pichler, A. J. Daley, and P. Zoller, Phys. Rev. A **82**, 063605 (2010).
[28] M. Spanner, I. Franco, and P. Brumer, Phys. Rev. A **80**, 053402 (2009).